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Solution of the advection equation using the iteration variation method

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ABSTRACT

Partial differential equations, particularly one-dimensional advection equations, frequently arise in various physics and engineering problems, including heat transfer and fluid flow. This study used the Iteration Variation method to solve the one-dimensional advection equation. The Iteration Variation method was chosen for its advantages in providing a solution that converges quickly without requiring complex linearisation. This study aims to analyse the effectiveness of the Iteration Variation method in producing analytical solutions and compare them with exact solutions for specific cases. The results show that the solutions obtained by the Iteration Variation method are close to the exact solution, with a relatively small error rate. Therefore, this method can be an efficient alternative for solving one-dimensional advection problems. The results of this study demonstrate that the Iteration Variation method can simplify the completion process and provide accurate results in the initial iteration, requiring only four iterations. The form of the solution is visualised using the MAPLE computer program. Although the complexity of the algebra increases with the number of iterations, it can be achieved with the help of the MaPLE program, which allows for the visualisation of the solution both in its algebraic formula and in its graphical representation.

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1. INTRODUCTION

The phenomenon of flow and transport is a natural process that is important to study because it significantly impacts various engineering applications. This phenomenon occurs in different physical situations, including heat transfer, chemical separation processes, fluid flow in porous media, the dispersion of contaminants in liquids, and the transport of small particles, such as pollutants, salts, sediments, and others, in shallow waters.

Advection is a transportation mechanism in which substances or heat move along with the flow of fluids, either in air, water, or other media. Advection is crucial in transferring heat and substances in rivers, ocean currents, and wind in the atmosphere. One example of advection is the movement of clouds, where the clouds always move in the direction of the wind. The movement of clouds is a critical process in the formation of rain, beginning with the formation of clouds through evaporation. This process involves the evaporation of water from the Earth's surface, such as seas, rivers, and lakes, due to heating by sunlight. Water vapour that rises into the atmosphere then experiences condensation, which is the change of steam into small ice particles that form clouds.

Other advection events include the transport of pollutants or sludge in rivers by the flow of water from upstream to downstream, the horizontal spread of heat that causes the surrounding air to become warm, and various transportation events. A mathematical approach is essential to understand the phenomenon of advection in a more in-depth and systematic way. One commonly used mathematical model is **the advection equation,** which is formally a form of **partial differential equation.** This equation describes the change in a physical property, such as the concentration, temperature, or velocity of a particle over time and space affected by the movement of a fluid at a certain speed. In simple one-dimensional form, the advection equation relates the rate of change of a quantity to the direction and velocity of the flow. The advection equation is an equation of first-order linear waves. It belongs to the class of hyperbolic differential equations that describe the transport mechanism of a gas or liquid in a specific direction [1].

The advection equation is a nonlinear partial differential equation. A nonlinear partial differential equation is an equation that has nonlinear terms. Nonlinear partial differential equations are complicated to solve exactly. Therefore, several studies have discussed the solution of nonlinear partial differential equations using semi-analytical methods. Semi-analytic methods that can be used to solve nonlinear partial differential equations include the Adomian Decomposition method, Perturbation Homotopy, and Iteration Variation method.

Various semi-analytical methods have been developed as the need to solve complex partial differential equations efficiently has grown. One of them is the Iteration Variation method, which was first introduced by Ji-Huan He in the late 1990s and has proven effective in solving various types of differential equations, both ordinary and partial, in linear and nonlinear forms. Furthermore, this iteration variation method was developed by Ji-Huan He in 2007, see [2][3].

This method is constructed as a correction function that contains a Lagrange multiplier. Unlike other semi-analytic methods, constructing the Iteration Variation method does not involve linearisation or minor interference.

Research on the Iteration Variation method has also been conducted by [4] to solve the Korteweg-de Vries Equation (KdV). This study explains solving the Korweteg-De Vries equation using the iteration variation method. This KDV equation is a partial differential equation that describes waves on the surface of shallow water. Research on the Iteration Variation method was also conducted by [5]. In 2020, this study described solving Nonlinear Parabolic Equations using the iteration variation method. In some cases, the advection equation models physical phenomena involving mass transport, heat transfer, or momentum transfer. Advection equations often involve complex or nonlinear boundary conditions. The iteration variation method offers an effective and practical approach to solving nonlinear equations, particularly when analytical solutions are challenging. In this paper, we discuss the solution of the Advection Equation using the Iteration Variation method.

2. METHOD

The phenomenon of flow and transport is a natural process that is important to study because it significantly impacts various engineering applications. This phenomenon occurs in different physical situations, including heat transfer, chemical separation processes, fluid flow in porous media, the dispersion of contaminants in liquids, and the transport of small particles, such as pollutants, salts, sediments, and others, in shallow waters.

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3. MATHEMATICAL MODELS

The form of the advection equation with the flow velocity u is

$$u_t + uu_x = x \tag{5}$$

with initial conditions

$$u(x, 0) = 2 \tag{6}$$

This equation shows the flow dynamics at a velocity that depends on the value of the solution u, and is influenced by external sources in the form of positional functions x.

4. RESULTS AND DISCUSSION

Equations (5) and (6) can be rewritten in the form

$$u_t + uu_x - x = 0 \tag{7}$$

with initial conditions

$$u(x,0) = 2$$

This form is necessary for the equation to be easily formulated in a variational form. Next, the variation iteration method is applied. Equation (7) will be converted into an iteration variation method by composing the correction function to

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda_2 \,[\left(\frac{\partial u_n(x,\psi)}{\partial \psi}\right) + \tilde{u}_n(x,\psi)(\frac{\partial \tilde{u}_n(x,\psi)}{\partial x}) - x]\, d\psi \qquad (8)$$

Equation (8) is designed with integration against time variables t, making it more suitable for partial differential equations that depend on the evolution of time. In this form, the derivative regarding time is expressed through the ψ, while derivatives of space remain explicit $\frac{\partial u}{\partial x}$.

The common form of the one-dimensional advection equation used is

$$u_t + uu_x - x = 0$$

In the equation, the tribe u_t is the dominant linear operator, i.e., the first derivative of time. Thus, based on the general formula of the Lagrange multiplier in the iteration variation method.

$$\lambda = \frac{(-1)^m}{(m-1)!}(s-t)^{m-1}$$

$$\lambda = \frac{(-1)^1}{(1-1)!}(s-t)^{1-1}$$

$$\lambda = -1$$

Next, substitute $\lambda_1 = -1$ into the equation (8), iterations of variations are obtained, namely.

$$u_{n+1}(x,t) \quad = \quad u_n(x,t) - \int_0^t [\left(\frac{\partial u_n(x,\psi)}{\partial \psi}\right) + \tilde{u}_n(x,\psi)(\frac{\partial \tilde{u}_n(x,\psi)}{\partial x}) - x]\, d\psi$$

with the initial value

$$u_0(x,t) = 2$$

for $n = 0$,

$$\begin{aligned} u_1(x,t) &= u_0(x,t) - \int_0^t [\left(\frac{\partial u_0(x,\psi)}{\partial \psi}\right) + \tilde{u}_0(x,\psi)(\frac{\partial \tilde{u}_0(x,\psi)}{\partial x}) - x]\, d\psi \\ &= 2 - \int_0^t [(0) + (2)(0) - x]\, d\psi \end{aligned}$$

$$= \quad 2 + \int_0^t x \, d\psi$$
$$= \quad 2 + xt$$

The form of the solution obtained from the first iteration is shown in **Figure 1**. **Figure 2** illustrates that the solution is linear in time, t, and proportional to the position, x, which is consistent with the initial solution characteristics of the advection equation. Not only do the images help to understand the shape of the solution, but they also give an initial idea of the convergence direction of the iteration variation method in the next iteration.

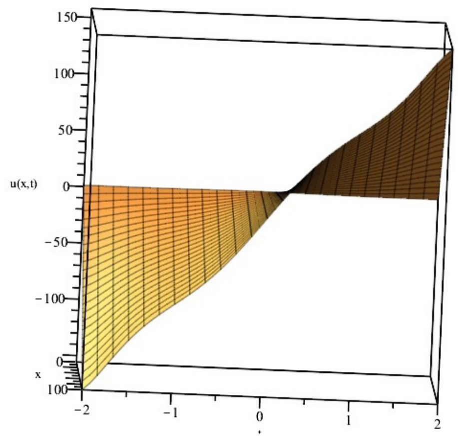

Figure 3. *Advection equation solution graph for iteration 1* ($n = 0$)

Next for n =1

$$u_2(x,t) = u_1(x,t) - \int_0^t [\left(\frac{\partial u_1(x,\psi)}{\partial \psi}\right) + \tilde{u}_1(x,\psi)(\frac{\partial \tilde{u}_1(x,\psi)}{\partial x}) - x] \, d\psi$$
$$= 2 + xt - \int_0^t [(x) + (2 + x\psi)(\psi) - x] \, d\psi$$
$$= 2 + xt - \int_0^t (2\psi + x\psi^2) \, d\psi$$
$$= 2 - t^2 + xt - \frac{xt^3}{3}$$

Figure 2 presents the results of the second iteration of the iteration variation method for solving the advection equation. The surface shape of the solution is approaching a smoother and more stable profile than in the first iteration. This demonstrates that iterative processes lead to convergent approaches to exact solutions. The change in the solution's contour from the first to the second iteration also enhances the effectiveness of the iteration variation method in solving partial differential equations semi-analytically.

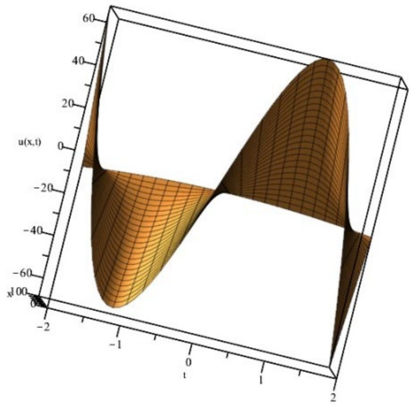

Figure 2 Advection equation solution graph *for* iteration 2 (n = 1)

For $n = 2$,

$$
\begin{aligned}
u_3(x,t) &= u_2(x,t) - \int_0^t [\left(\frac{\partial u_2(x,\psi)}{\partial \psi}\right) + \tilde{u}_2(x,\psi)(\frac{\partial \tilde{u}_2(x,\psi)}{\partial x}) - x]\, d\psi \\
&= 2 - t^2 + xt - \frac{xt^3}{3} - \int_0^t [(-2\psi + x - x\psi^2) + (2 - \psi^2 + x\psi - \frac{x\psi^3}{3})(\psi - \frac{\psi^3}{3}) - x]\, d\psi \\
&= 2 - t^2 + xt - \frac{xt^3}{3} - \int_0^t [-\frac{5\psi^3}{3} - \frac{2x\psi^4}{3} + \frac{\psi^5}{3} + \frac{x\psi^6}{9})]\, d\psi \\
&= 2 - t^2 + xt - \frac{xt^3}{3} - [-\frac{5t^4}{12} - \frac{2xt^5}{15} + \frac{t^6}{18} + \frac{xt^7}{63}] \\
&= 2 - t^2 + \frac{5t^4}{12} - \frac{t^6}{18} + xt - \frac{xt^3}{3} + \frac{2xt^5}{15} - \frac{xt^7}{63}
\end{aligned}
$$

Figure 3 shows a graph of the results from completing the third iteration of the iteration variation method for the advection equation. In this iteration, the solution form is more complex algebraically, but it is getting closer to the exact solution form. This can be observed from the emergence of high-ranking tribes at time variable t and position x, which indicates that this method captures the nonlinear dynamics of the equation gradually and systematically.

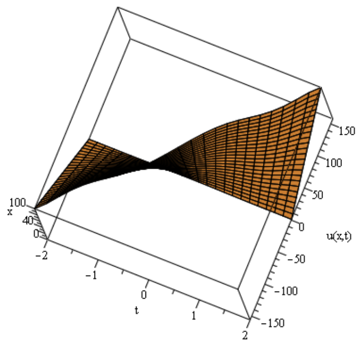

Figure 3. Advection equation solution graph *for* iteration 3 (n = 2)

Furthermore,
For $n = 3$

$$
\begin{aligned}
u_4(x,t) &= u_3(x,t) - \int_0^t [\left(\frac{\partial u_3(x,\psi)}{\partial \psi}\right) + \tilde{u}_3(x,\psi)(\frac{\partial \tilde{u}_3(x,\psi)}{\partial x}) - x]\, d\psi \\
&= 2 - t^2 + \frac{5t^4}{12} - \frac{t^6}{18} + xt - \frac{xt^3}{3} + \frac{2xt^5}{15} - \frac{xt^7}{63} \\
&\quad - \int_0^t [\left(-2\psi + \frac{5\psi^3}{3} - \frac{\psi^5}{3} + x - x\psi^2 + \frac{2x\psi^4}{3} - \frac{x\psi^6}{9}\right) + (2 - \psi^2 \\
&\quad + \frac{5\psi^4}{12} - \frac{\psi^6}{18} + x\psi - \frac{x\psi^3}{3} + \frac{2x\psi^5}{15} - \frac{x\psi^7}{63})(\psi - \frac{\psi^3}{3} + \frac{2\psi^5}{15} - \frac{\psi^7}{63}) \\
&\quad - x]\, d\psi \\
&= 2 - t^2 + \frac{5t^4}{12} - \frac{t^6}{18} + xt - \frac{xt^3}{3} + \frac{2xt^5}{15} - \frac{xt^7}{63} \\
&\quad - \int_0^t [\left(\frac{41\psi^5}{60} + \frac{4x\psi^6}{15} - \frac{12231\psi^7}{15(63)36} - \frac{342x\psi^8}{63(45)} + \frac{55080\psi^9}{12(15)(63)18(3)} \right.\\
&\quad + \frac{1206x\psi^{10}}{15(15)3(63)} - \frac{2862\psi^{11}}{12(63)18(15)} - \frac{4x\psi^{12}}{15(63)} + \frac{\psi^{13}}{18(63)} \\
&\quad \left. + \frac{x\psi^{14}}{63(63)}\right)]\, d\psi \\
&= 2 - t^2 + \frac{5t^4}{12} - \frac{t^6}{18} + xt - \frac{xt^3}{3} + \frac{2xt^5}{15} - \frac{xt^7}{63} - [\frac{41t^6}{60(6)} + \frac{4xt^7}{15(7)} - \frac{12231t^8}{15(63)36(8)} \\
&\quad - \frac{342xt^9}{63(45)9} + \frac{55080t^{10}}{12(15)(63)18(3)10} + \frac{1206xt^{11}}{15(15)3(63)11} \\
&\quad - \frac{2862t^{12}}{12(63)18(15)12} - \frac{4xt^{13}}{15(63)13} + \frac{t^{14}}{18(63)14} + \frac{xt^{15}}{63(63)15}] \\
&= 2 - t^2 + \frac{5t^4}{12} - \frac{t^6}{18} - \frac{41t^6}{60(6)} + \frac{12231t^8}{15(63)36(8)} - \frac{55080t^{10}}{12(15)(63)18(3)10} \\
&\quad + \frac{2862t^{12}}{12(63)18(15)12} - \frac{t^{14}}{18(63)14} + xt - \frac{xt^3}{3} + \frac{2xt^5}{15} - \frac{xt^7}{63} \\
&\quad - \frac{4xt^7}{15(7)} + \frac{342xt^9}{63(45)9} - \frac{1206xt^{11}}{15(15)3(63)11} + \frac{4xt^{13}}{15(63)13} - \frac{xt^{15}}{63(63)15}
\end{aligned}
$$

Figure 4. Displays a graph of the completion results of the fourth iteration of the iteration variation method on the advection equation. For the next iteration, the third iteration, a more complex form of solution is obtained, but it is getting closer to the exact form of the solution. This can be observed from the emergence of high-ranking tribes at time variable t and position x, which indicates that this method captures the nonlinear dynamics of the equation in a gradual and systematic manner. A visualisation of the results from the third iteration is shown in **Figure 4.**

The formula form of iteration 4 expresses the Taylor series form of the function.

$$u = 2.\operatorname{sech}(t) + x.\tanh(t)$$

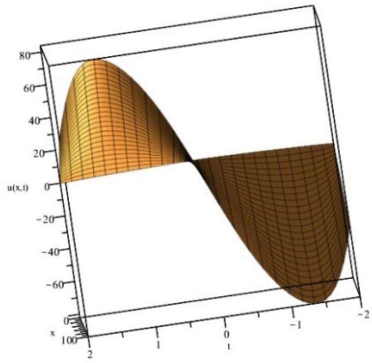

Figure 4. Advection equation solution graph *for* iteration 4 (n = 3)

$u = 2.\operatorname{sech}(t) + x.\tanh(t)$ is the exact solution of the advection equation (7)

To get a more accurate solution approach, the iteration process continues until the fourth iteration. Each iteration yields a more complex solution form, yet it is closer to the behavioural characteristics of the exact solution. Successively, the results of the first to fourth iterations form a series of solutions that converge on the precise form of the advection equation. The graphs of the iteration results are shown in Figures 1 through 4. In the first iteration graph, the surface shape of the solution still shows a linear change. In contrast, in the second iteration graph, a curvature begins to appear, indicating the nonlinear effect of the advection equation. The third and fourth iteration images demonstrate that the iteration variation method provides a practical and rapidly converging solution approach.

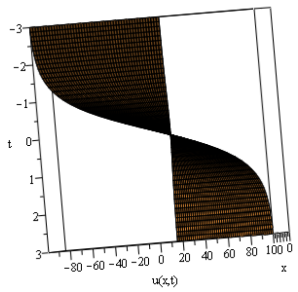

Figure 5. Graph of the exact solution of the advection equation

5. CONCLUSION

In this paper, we have discussed the solution of the advection equation using the variation iteration method. The discussion results show that the iteration variation method provides a solution approach in the form of a series that converges to the exact solution. The iteration process can deliver results close to the real solution in just a few initial steps, making it computationally efficient. The simulation results also show that the solution of the iteration variation method for advection cases is close to the exact solution with a small error rate, especially in the 4th and 5th iterations. This shows that this method is reliable and practical enough to solve nonlinear partial differential equations. Based on simulations and the results of iterative solutions, the convergence of the solutions becomes increasingly evident as the number of iterations increases, making this method suitable for similar cases with specific initial conditions.

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REFERENCES

[1] M. Abdy, M. S. Wahyuni, and N. F. Awaliyah, "Solusi Persamaan Adveksi-Difusi dengan Metode Dekomposisi Adomian Laplace," *J. Math. Comput. Stat.*, vol. 5, no. 1, p. 40, 2022, doi: 10.35580/jmathcos.v5i1.32249.

[2] J. H. He, "Variational iteration method-a kind of non-linear analytical technique: some examples.," *Int. 1207 J. Non-Linear Mech.*, vol. 34, no. 4, pp. 699–708, 1999.

[3] J. H. He, "Variational iteration method-Some recent results and new interpretations," *J. Comput. Appl. Math.*, vol. 207, no. 1, pp. 3–17, 2007, doi: 10.1016/j.cam.2006.07.009.

[4] D. Maulina, "PENYELESAIAN PERSAMAAN KDV (KORTEWEG DE VRIES)DENGAN METODE ITERASI VARIASI," pp. 1–23, 2019.

[5] W. Wartono and I. Suryani, "The solution of nonlinear parabolic equation using variational iteration method," *J. Mat. Stat. dan Komputasi*, vol. 16, no. 3, p. 287, 2020, doi: 10.20956/jmsk.v16i3.8468.