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A theoretical analysis of Hyper-Wiener indices in graphs derived from algebraic structures

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ABSTRACT

The Hyper-Wiener index is a widely used topological descriptor that quantifies the structural complexity of graphs, particularly those arising from algebraic structures. This paper presents a structured synthesis of key theorems related to the Hyper-Wiener index in coprime graphs, non-coprime graphs, and power graphs constructed from the integer modulo group and the dihedral group. Adopting a systematic literature review approach, we compile and restate formal results, including explicit formulas and proven properties. Each theorem is analyzed in relation to the algebraic structure of its underlying group and the resulting graph topology. Our findings highlight how group-theoretic properties—such as order, operation, and element interactions—directly impact the Hyper-Wiener index. This survey is intended to support researchers by providing a conceptual bridge between group theory and topological graph theory, and by identifying potential directions for future work.

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1. INTRODUCTION

Research in topological indices constitutes a vital branch of mathematical chemistry and graph theory. A topological index is a numerical descriptor that effectively captures the overall topology of a molecular or graph structure, serving as a critical component in **Quantitative Structure-Activity/Property Relationship (QSAR/QSPR)** modeling [1]. The classical **Wiener Index** ($W(G)$) is the oldest and most widely used descriptor, defined as the sum of the shortest distances between all pairs of vertices in a graph.

Despite its efficacy, $W(G)$ often lacks the necessary sensitivity to distinguish complex or highly branched graph structures [2]. To address this limitation, the **Hyper-Wiener Index** ($WW(G)$) was introduced as a robust generalization [3]. The Hyper-Wiener index quantifies the structural complexity of a graph by summing the pairwise distances and their squares. Formally, $WW(G)$ is defined as

$$WW(G) = \left(\sum_{\{x,y\} \subseteq V(G)} d(x,y) + d^2(x,y) \right)$$

Beyond its original applications in chemical graph theory, this index provides a deeper understanding of graph structure, derived directly from underlying algebraic structures [4].

In the field of Algebraic Graph Theory, various graphs are constructed by using the elements of an algebraic group as vertices, with adjacency rules based on specific algebraic properties or element interactions. This approach facilitates the visual and structural analysis of the internal properties of groups [5].

This paper focuses on three significant types of group-derived graphs, where the underlying group properties are critical for defining vertex adjacency:

- **Coprime Graphs (Γ_G):** Two distinct vertices, x and y, are connected by an edge if and only if the greatest common divisor of their orders, $\gcd(ord(x), ord(y))$, is equal to 1 [6].
- **Non-Coprime Graphs ($\overline{\Gamma_G}$):** These are the complements of coprime graphs, where two distinct non-identity elements are connected if their orders share a common divisor greater than one, $\gcd(ord(x), ord(y)) \neq 1$ [7].
- **Power Graphs:** An edge connects two distinct vertices, a and b, if one can be expressed as a positive integer power of the other (i.e., $a = b^x$ or $b = a^y$ for $x, y \in \mathbb{Z}^+$) [8].

This study examines explicitly two fundamental algebraic structures, whose properties are detailed in standard texts [5]: the **Group of Integers Modulo n ($\mathbb{Z}_n$)** and the **Dihedral Group (D_{2n})**. The group $\mathbb{Z}_n$ (with addition modulo n) serves as the primary example of a finite **Cyclic and Abelian Group**. Conversely, the Dihedral Group D_{2n} (noting the standard convention for a group of n elements) is a classic and foundational example of a finite **Non-Abelian Group**.

The deliberate selection of these two algebraically distinct groups—one commutative and one non-commutative—is crucial for a comprehensive, comparative analysis. By focusing on $WW(G)$ derived from both $\mathbb{Z}_n$ and D_{2n}, we can explore how core group properties, such as element order, generating elements, and the presence or absence of commutativity, structurally influence the resulting graph topology and, consequently, the value of the Hyper-Wiener index.

Several researchers have successfully developed explicit formulas for calculating the Hyper-Wiener index for these group-derived graphs under specific algebraic conditions. For instance, the $WW(G)$ formula has been established for Power Graphs derived from the integer group modulo $\mathbb{Z}_n$ [9]. Similarly, results exist for the Coprime Graph of D_{2n} under specific conditions, such as when n is a prime power [10]. Other works have focused on Non-Coprime Graphs and various parameterizations of n.

However, despite these crucial individual findings, the existing literature necessitates a structured, synthetic, and comprehensive review. The primary research gap addressed by this paper is the absence of a unified framework that compiles, reformulates, and comparatively analyzes all proven results for the Hyper-Wiener index across all three graph types (coprime, non-coprime, and power graphs), derived from both $\mathbb{Z}_n$ and D_{2n} groups, under their various parameter conditions ($n = p$, $n = 2^k$, and $n = p^k$).

This article aims to review and synthesize the extant theorems related to the Hyper-Wiener index in coprime, non-coprime, and power graphs derived from the integer modulo group and the dihedral group. Our work adopts a systematic literature review approach to organize previously published results and analyze them from a structured algebraic and topological perspective.

By providing this systematic review, we hope to enhance the understanding of the direct connection between group-theoretic properties and topological graph invariants. This survey is intended to serve as a conceptual bridge for researchers [11], offering a comprehensive reference and facilitating the identification of potential directions for future comparative work in algebraic graph theory[12][13].

2. METHOD

The methodology employed in this paper is a Systematic Literature Review (SLR). The primary goal of this approach is not to generate new empirical results or original theorems, but rather to identify, evaluate, and synthesize all high-quality scholarly research relevant to a specific research question [14]. In this context, the focus is on the explicit formulas and properties of the Hyper-Wiener Index ($WW(G)$) derived from specific algebraic graph constructions.

2.1 Search Strategy and Data Sources

The literature search was conducted across several leading academic databases, including Scopus, Web of Science, ScienceDirect, and Google Scholar, to ensure comprehensive coverage of published works in graph theory and abstract algebra. The search strategy involved combining keywords related to the index being studied with the specific graph and group structures:

- Topological Index Keywords: "Hyper-Wiener Index", "$WW(G)$", "Wiener Index".
- Graph Construction Keywords: "Coprime Graph", "Non-Coprime Graph", "Power Graph".
- Algebraic Structure Keywords: "Dihedral Group D_{2n}", "Integer Modulo Group $\mathbb{Z}_n$.

The search queries utilized Boolean operators (AND, OR) to narrow the results, such as: ("Hyper-Wiener Index" AND "Coprime Graph" AND "Dihedral Group"). Initial search results include publications to capture all relevant historical and current findings.

2.2. Inclusion and Exclusion Criteria

The selection of articles for synthesis was based on stringent criteria to ensure methodological relevance:

1. **Inclusion Criteria:**
   - Articles must explicitly provide a **formal, proven theorem or formula** for the Hyper-Wiener Index ($WW(G)$).

- The underlying graph must be one of the three focused types: Coprime, Non-Coprime, or Power Graphs.
- The group from which the graph is derived must be $\mathbb{Z}_n$ or D_{2n}.
- Articles must be peer-reviewed publications (journal articles or conference proceedings) [15].

2. **Exclusion Criteria:**
   - Articles discussing only other topological indices (e.g., Randić, Szeged, or Balaban Index) without reference to $WW(G)$.
   - Studies on $\mathbb{Z}_n$ or D_{2n} that do not involve graph construction or the calculation of indices.
   - Theses, dissertations, or unpublished preprints.
   - Articles that provide only computational results without a formal mathematical proof.

2.3. Data Extraction and Synthesis

The selected articles were systematically reviewed for data extraction. The extracted data included:

- The specific algebraic group ($\mathbb{Z}_n$ or D_{2n}) and its parameter conditions ($n = p$, $n = 2^k$, etc.).
- The type of graph construction (Coprime, Non-Coprime, or Power Graph).
- The exact formal statement of the theorem (including Lemmas necessary for proof construction).
- The explicit formula derived for $WW(G)$.

The final stage involved **synthesis**, where the compiled theorems were restated and organized into a unified structure within the Results and Discussion section. This organization facilitated the direct comparison and analysis of the topological consequences arising from distinct algebraic properties of the groups [14].

3. RESULTS AND DISCUSSION

To initiate the analysis, we first define the Hyper-Wiener index, which plays a fundamental role in measuring the structural properties of graphs based on vertex distances. This concept serves as the foundation for the subsequent results and is formally presented as follows

Definition 3.1. [16] Let $G = (V, E)$ be a connected graph with vertex set V. The Hyper-Wiener index of a graph G denoted by $WW(G)$, is defined as,

$$WW(G) = \frac{1}{2}\left(\sum_{\{x,y\}\subseteq V(G)} d(x,y) + d^2(x,y)\right)$$

where $d(x,y)$ is the distance from vertex x to y.

After introducing the Hyper-Wiener index, we now turn to the algebraic structures used to construct the graphs under study, starting with the group of integers modulo n.

Definition 3.2. [5] For any positive integer n, the set $\mathbb{Z}^+$ can be partitioned into n residue classes modulo n, corresponding to the remainders $0, 1, 2, \dots, n-1$ when divided by n.

By equipping the set of residue classes modulo n with addition modulo n, a group known as the integers modulo n, denoted by $\mathbb{Z}_n$ is formed.

$$\mathbb{Z}_n = \{0,1,2, \dots, n-1\}.$$

With an understanding of modular arithmetic and the group structure it induces, we are now prepared to introduce the dihedral group, a more intricate algebraic object used in the subsequent analysis.

Definition 3.3. [5] The dihedral group, denoted by D_n, is the group of symmetries of a regular n-gon. It consists of n rotations and n reflections, for a total of $2n$ elements. The group operation is a composition of symmetries.

$$D_n6 = \langle x, y | x^n = e, y^2 = e, yxy^{-1} = x^{-1} \rangle.$$

Once the dihedral group has been defined, we proceed to construct several types of graphs based on its elements. One such graph is the coprime graph, defined as follows.

Definition 3.4. [6] The coprime graph, denoted by Γ_G, is an undirected graph where each vertex represents an element of the group G, and two distinct vertices x and y are connected by an edge if and only if the greatest common divisor of their orders, gcd $(|x|, |y|)$, is equal to 1.

Example 3.1. Let $\mathbb{Z}_5 = \{0,1,2,3,4\}$ be the additive group modulo 5. We construct the coprime graph $\Gamma_{\mathbb{Z}_5}$ as follows

Table 3.1. Order Element of $\mathbb{Z}_5$

Element	Order
0	1
1	5
2	5
3	5
4	5

Based on Definition 3.4, after calculating the order of each element in $\mathbb{Z}_5$, we can then construct the coprime graph of $\mathbb{Z}_5$ as follows

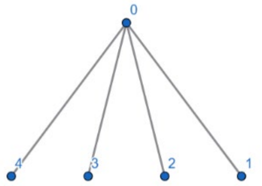

Figure 3.1. Coprime Graph of $\mathbb{Z}_5$

In addition to coprime graphs, it is also helpful to consider their complementary structure, namely the non-coprime graph, defined as follows.

Definition 3.5. [7] Consider a finite group G. The non-coprime graph associated with G, denoted by Γ_G, is constructed by taking all non-identity elements of G as its vertices. In this graph, two distinct vertices x and y are connected by an edge if and only if the orders of x and y share a common divisor greater than one, i.e., $\gcd(|x|,|y|) \neq 1$.

Example 3.2. Let $\mathbb{Z}_6 = \{0,1,2,3,4,5\}$ be the additive group modulo 6. We construct the non-coprime graph $\bar{\Gamma}_{\mathbb{Z}_5}$ as follows

Table 3.2. Element Order of $\mathbb{Z}_6$

Element $\mathbb{Z}_6$	Order
1	6
2	3
3	2
4	4
5	6

Based on Definition 3.5, after calculating the order of each element in $\mathbb{Z}_6$, we can then construct the non-coprime graph of $\mathbb{Z}_6$ as follows.

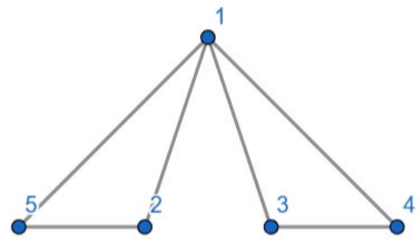

Figure 3.2. Non-coprime Graph of $\mathbb{Z}_6$

While coprime and non-coprime graphs focus on number-theoretic properties, the following structure emphasizes algebraic interaction within groups.

Definition 3.6. [8] A power graph of a group G is defined as a graph where each vertex represents an element of G, and an edge connects two distinct vertices a and b if one of them can be expressed as a positive integer power of the other; that is, there exist $x, y \in \mathbb{Z}^+$such that $a = b^x$ or $b = a^y$.

Example 3.3 Let $\mathbb{Z}_3 = \{0,1,2\}$ be the additive group modulo 3. It contains three elements with addition defined modulo 3.

Table 3.3. Neighbor Relationships in $\mathbb{Z}_3$

Pair of Elements	**Power Relation**
(0, 1)	$1^3 = 0$
(0, 2)	$2^3 = 0$
(2,1)	$1^2 = 0$

Based on the adjacency information in Table 3.3, the resulting graph is shown in Figure 3.3

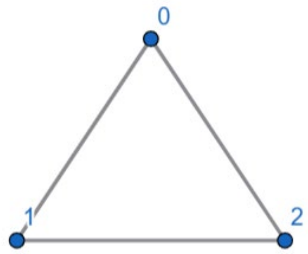

Figure 3.3. Power Graph of $\mathbb{Z}_3$

With the foundational definitions of various graph structures in place, we now proceed to analyze their properties in the context of specific algebraic groups, starting with the following lemma.

Lemma 3.1. [17] Consider the dihedral group D_{2n} with $n \geq 3$. When n is an odd prime, the degree of a vertex in the coprime graph corresponding to D_{2n} can be determined as follows

a. $\deg(a^i) = n + 1, \forall i \in \mathbb{Z}, 1 \leq i \leq n$
b. $\deg(a^i b) = n, \forall i \in \mathbb{Z}, 1 \leq i \leq n$
c. $\deg(e) = 2n - 1$.

Lemma 3.2. [10] Let D_{2n} be a dihedral group with $n \geq 3$. if $n = 2^k, k \in \mathbb{N}$, then the coprime graph formed from the dihedral group D_{2n} is a complete bipartite graph.

Lemma 3.3. [18] The non-coprime graph of the modulo n integer group for $n = p^m$, where p is a prime number and m is a positive integer, is a complete graph $K_{(n-1)}$.

Lemma 3.4. [19] Given a group of integers modulo n ($\mathbb{Z}_n$) with operation $+_{mod(n)}$. if $n = p^k$ for all $k \in \mathbb{N}$ and p is a prime number, then the power graph of the integer group $\mathbb{Z}_n$ is the complete graph K_n .

Lemma 3.5. [9] If $n = p^m$ where p is a prime number and m is a natural number, then the power graph of the dihedral group D_{2n} contains K_n and $K_{1,n}$ as subgraphs, with $V(K_n) \backslash \cap V(K_{1,n}) = \{e\}$.

Based on the previous definitions and lemmas, we now begin formulating the Hyper-Wiener index of the coprime graph of the dihedral group in the case where $n = p$, with p being an odd prime number

Theorem 3.1. [17] Let Γ_{D2n} represent the coprime graph of a dihedral group. When $n = p$, where p is an odd prime number, the Hyper-Wiener index of this coprime graph can be determined accordingly.

$$WW(\Gamma_{D2n}) = n^2 - 5n + 2.$$

Proof. The coprime graph has 3 Partitions, namely: $V_1 = \{e\}, V_2 = \{a, a^2, \dots, a^{n-1}\}$, and $V_3 = \{b, ab, a^2 b, \dots, a^{n-1} b\}$. Based on Lemma 3.1, we get $deg(v_1) = 2n - 1, deg(v_2) =$

$n + 1$, and $deg(v_3) = n$, for $v_1 \in V_1, v_2 \in V_2$ and $v_3 \in V_3$. Let any $u_1, v_1 \in V_1$, $u_2, v_2 \in V_2$ and $u_3, v_3 \in V_3$ thus, the $WW(\Gamma_{D2n})$ as follows

$$WW(\Gamma_{D2n}) = \frac{1}{2}\left(\sum_{u,v\in V(G)} d(u,v) + d^2(u,v)\right)$$
$$= \frac{1}{2}\left[\left(\sum_{u_1,v_2\in V(G)} d(u_1,v_2) + d^2(u_1,v_2)\right) + \left(\sum_{u_1,v_3\in V(G)} d(u_1,v_3) + d^2(u_1,v_3)\right)\right.$$
$$+ \left(\sum_{u_2,v_3\in V(G)} d(u_2,v_3) + d^2(u_2,v_3)\right) + \left(\sum_{u_2,v_2\in V(G)} d(u_2,v_2) + d^2(u_2,v_2)\right)$$
$$\left.+ \left(\sum_{u_3,v_3\in V(G)} d(u_3,v_3) + d^2(u_3,v_3)\right)\right]$$

Based on the definition of the dihedral group and the Lemma, we derive the following results regarding the number of vertex pairs in various vertex classes:

Table 3.4. Number of Vertex Pairs Between Classes

Vertex Class Pair	Power Relation
$V(v_1, v_2)$	$n - 1$
$V(v_1, v_3)$	n
$V(v_2, v_3)$	$n(n-1)$
$V(v_2, v_2)$	$\frac{1}{2}(n^2 - 3n + 2)$
$V(v_3, v_3)$	$\frac{1}{2}n(n-1)$

Therefore

$$WW(\Gamma_{D2n}) = \frac{1}{2}[(2(n-1)) + (2(n)) + ((n^2 - n)2) + ([(n^2 - 2n) - (2 + 3 + \cdots$$
$$+ (n-1)]6) + ([(n^2 - n) - (1 + 2 + \cdots + (n-1)]6]$$
$$= \frac{1}{2}\left[\left((2n^2 - 2n - 2) + \left(\frac{1}{2}(n^2 - 3n + 2)6\right) + \left(\frac{1}{2}n(n-1)6\right)\right)\right]$$
$$= \frac{1}{2}(8n^2 - 10n + 4)$$
$$= n^2 - 5n + 2 \ \blacksquare$$

Having formulated the Hyper-Wiener index of the coprime graph of the dihedral group for $n = p$, where p is an odd prime, we now proceed to examine the case where $n = 2^k$, with k being a positive integer.

Theorem 3.2. [17] Let Γ_{D2n} represent the coprime graph of the dihedral group. When $n = 2^k$ for $k \in \mathbb{N}$, the Hyper-Wiener index of this graph is given by

$$WW(\Gamma_{D2n}) = 6n^2 - 7n + 2.$$

Proof. If $n = 2^k$ for some $k \in \mathbb{N}$, we obtain a coprime graph with 2 partitions, namely $V_1 = \{e\}$,$V_2 = \{a, a^2, a^3, b, ab, a^2b, a^3b\}$. Based on Lemma 3.2, we have $deg(v_1) = 2n - 1$, $deg(v_2) = 1$, Let any $u_1, v_1 \in V_1,\ u_2, v_2 \in V_2$ so the $WW(\Gamma_{D2n})$ is

$$WW(\Gamma_{D_{2n}}) = \frac{1}{2}\left(\sum_{u,v\in\Gamma_{D_{2n}}} d(u,v) + d^2(u,v)\right)$$
$$= \frac{1}{2}\left[\left(\sum_{v_1,v_2\in V(\Gamma_{D_{2n}})} d(v_1,v_2) + d^2(v_1,v_2)\right) + \left(\sum_{u_2,v_2\in V(\Gamma_{D_{2n}})} d(u_2,v_2) + d^2(u_2,v_2)\right)\right]$$

Based on the definition of the dihedral group, it is obtained that: the number of vertex pairs in $V(v_1, v_2) = 2n - 1$, the number of vertex pairs in $V(u_2, v_2) = 2n^2 - 3n + 1$ therefore,

$$\begin{aligned}
WW(\Gamma_{D_{2n}}) &= \frac{1}{2}\left[\left((2(n-1))2\right) + \left([(2n^2 - 2n) - (2 + 3 + \cdots + (2n-1))6]\right)\right] \\
&= \frac{1}{2}[2(n-1)2 + (2n^2 - 3n + 1)6] \\
&= \frac{1}{2}[(4n-2) + (12n^2 - 18n + 6)] \\
&= \frac{1}{2}(12n^2 - 14n + 4) \\
&= 6n^2 - 7n + 2 \ \blacksquare
\end{aligned}$$

Having examined the Hyper-Wiener index of the coprime graph associated with the dihedral group, we now turn our attention to the non-coprime graph constructed from the group of integers modulo n, specifically in the case where $n = p^k$, with p being a prime number and $k \in \mathbb{N}$.

Theorem 3.3. [20] Given $\mathbb{Z}_n$ with n being a prime power, the Hyper-Wiener index of the non-coprime graph of $\mathbb{Z}_n$ is $WW(\bar{\Gamma}_{\mathbb{Z}_n}) = \frac{1}{2}(n-1)(n-2)$.

Proof. Based on Lemma 3.3, since $\bar{\Gamma}_{\mathbb{Z}_n}$ is a complete graph $K_{(n-1)}$, for any $x, y \in V(\bar{\Gamma}_{\mathbb{Z}_n})$, $d(x,y) = 0$ if $x = y$ and $d(x,y) = 1$ if $x \neq y$. Thus, $\frac{1}{2}[d(x,y) + d^2(x,y)] = 1$ if $x \neq y$. Furthermore, since $\bar{\Gamma}_{\mathbb{Z}_n}$ has $n-1$ distinct vertices, the $WW(\Gamma_{D2n})$is

$$WW\left(\bar{\Gamma}_{\mathbb{Z}_n}\right) = \frac{1}{2}\ (n-1)\ (n-2).$$

Having examined the Hyper-Wiener index of the non-coprime graph for $n = p^k$, we now consider the case where n is the product of two distinct prime numbers, $n = pq$

Theorem 3.4 [20] Let $\mathbb{Z}_n$ be the modulo integer group with $n = p_1p_2$, where p_1 and p_2 are two distinct prime numbers. Then the hyper-Wiener index of the non-coprime graph of $\mathbb{Z}_n$ is

$WW\left(\bar{\Gamma}_{\mathbb{Z}_n}\right) = \frac{1}{2}\left[(p_2 - 1)(p_2 - 2) + (p_1 - 1)(p_1 - 2) + (n - p_2 - p_1)(n - p_2 - p_1 + 1)\right] + (n - p_2 - p_1 + 1)(p_1 + p_2 - 2) + 3(p_1 - 1)(p_2 - 1).$

Proof. See refer to [20]

After analyzing the Hyper-Wiener index of the non-coprime graph, we now shift to a different type of graph, namely the power graph constructed from the group of integers modulo n where $n = p^k$

Theorem 3.5. [9] Consider the power graph $\Gamma_{\mathbb{Z}_n}$, where n is a power of prime number, i.e., $n = p^k$ for some positive integer k and prime p. The vertex set corresponds to the elements of the additive group of integers modulo n. The Hyper-Wiener index of $\Gamma_{\mathbb{Z}_n}$ is

$$WW\left(\Gamma_{\mathbb{Z}_n}\right) = \frac{n(n-1)}{2}.$$

Proof. According to Lemma 3.4 $\Gamma_{\mathbb{Z}_n}$ is a complete graph, thus $d(x,y) = 1, \forall x,y \in V(\Gamma_{\mathbb{Z}_n})$. Furthermore, since this is a complete graph, the number of edges is the combination of n choose 2. That is, as follows

$$\begin{aligned} C_2^n &= \frac{n!}{2!\,(n-2)!} \\ &= \frac{n(n-1)(n-2)!}{2!\,(n-2)!} \\ &= \frac{n(n-1)}{2} \ \blacksquare \end{aligned}$$

Having formulated the Hyper-Wiener index for the power graph on the group of integers modulo n, we now turn our attention to the power graph derived from a different algebraic structure the dihedral group

Theorem 3.6. [3] Suppose $\Gamma_{D_{2n}}$ is a power graph on the dihedral group with $n \geq 3$. The Hyper-Wiener index of $\Gamma_{D_{2n}}$ is $WW\left(\Gamma_{D_{2n}}\right) = 5n^2 - 4n$.

Proof. Based on Lemma 3.5, the vertices in the power graph of the group $\Gamma_{D_{2n}}$ can be partitioned into 3 parts as follows

$$\begin{aligned} V_1 &= \{a, a^2, a^3, \dots, a^{n-1}\} \\ V_2 &= \{b, ab, a^2b, a^3b, \dots, a^{n-1}b\} \\ V_3 &= \{e\} \end{aligned}$$

Based on the three partitions and Lemma 3.5, the problem in the proof can be divided into 5 cases:
Case 1: $d(x,y) = 1, \forall x,y \in V_1$, Case 2: $d(x,y) = 2, \forall x,y \in V_2$, Case 3: $d(x,y) = 1, \forall x \in V_1, \forall y \in V_3$, Case 4: $d(x,y) = 1, \forall x \in V_2, \forall y \in V_3$, Case 5: $d(x,y) = 2, \forall x \in V_1, \forall y \in V_2$

Thus, the Hyper-Wiener index of $\Gamma_{D_{2n}}$ is obtained as follows

$$\begin{aligned} WW(\Gamma_{D_{2n}}) &= \frac{1}{2}C_2^{n-1}(1+1^2) + \frac{1}{2}C_2^{n}(2+2^2) + \frac{1}{2}(n-1)(1)(1+1^2) \\ &\quad + \frac{1}{2}(n)(1)(1+1^2) + \frac{1}{2}(n-1)(n)(2+2^2) \\ &= \frac{1}{2}(n^2-3n+2+3n^2-3n+2n-2+2n+6n^2-6n \\ &= 5n^2-4n \ \blacksquare \end{aligned}$$

4. CONCLUSION

In this study, we have explored the computation of the Hyper-Wiener index for several classes of graphs derived from algebraic structures, including coprime graphs, non-coprime graphs, and power graphs. Specifically, we examined these graphs on groups such as the group of integers modulo n and dihedral groups under various conditions of n, such as when $n = p, n = 2^k$ and $n = p^k$, with p being a prime number. The results show how the structure of the underlying group significantly influences the properties and values of the Hyper-Wiener index. These findings contribute to a deeper understanding of graph invariants in algebraic graph theory and may support future research in chemical graph theory or network topology. Further studies may extend this analysis to other graph indices or more complex group structures.

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