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Analysis of the ABC Index on Inverse and Non-Commuting Graphs of Finite Groups

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ABSTRACT

This study investigates the Atom Bond Connectivity (ABC) index applied to inverse and non-commuting graphs of finite groups. The research focuses on two distinct structures: inverse graphs formed from generalized quaternion groups and non-commuting graphs derived from non-abelian finite groups. By utilizing topological index Theory, particularly the ABC index introduced by Estrada, the study aims to characterize the graph structures based on vertex degrees. For the inverse graphs of generalized quaternion groups, the ABC index is determined using known group properties and edge classifications. Similarly, the ABC index of non-commuting graphs is explored by analyzing group centers and the connectivity patterns among non-central elements. The results reveal that the ABC index provides a meaningful quantification of group connectivity, enhancing the understanding of structural properties in algebraic graph representations. This analysis contributes to the broader application of graph Theory in abstract algebra and theoretical chemistry.

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1. INTRODUCTION

A group is one of the fundamental topics in mathematics, particularly in abstract algebra. A group consists of a non-empty set G with a binary operation satisfying four axioms: closure, associativity, identity element, and the existence of inverses for all elements [1]. One approach to studying group structure is through graph Theory, by representing group elements as vertices and defining edges based on specific relations between those [2].

One example is the inverse graph, which is constructed from group elements that are not self-inverse, where two vertices are connected if their product is the identity element [3]. Another is the non-commuting graph, which consists of elements not in the center of the group; two vertices in this graph are connected if their elements do not commute [4].

In graph Theory, a topological index is a numerical value derived from the structure of the graph that represents characteristics such as vertex connectivity, edge count, and inter-vertex distances. These indices are useful in predicting structural properties in fields such as chemistry [5]. One widely studied topological index in recent years is the Atom Bond Connectivity (ABC) index, introduced by Estrada in 1998 [6]. The ABC index characterizes atom and bond interactions within molecular graphs and has been found to correlate well with molecular stability, bond energy, and chemical reactivity [7].

In addition to its applications in chemistry, the ABC index has been increasingly studied in the context of algebraic graph Theory. The study of the ABC index on graphs constructed from group structures provides insights into the internal relationships between group elements, particularly in terms of connectivity and symmetry. For example, in the case of generalized quaternion groups, the inverse graph exhibits a regular structure that simplifies the computation of the ABC index [8]. Similarly, non-commuting graphs derived from non-abelian groups often form multipartite graphs with a constant diameter and uniform eccentricity, offering unique properties that are reflected through their ABC indices [9].

The idea of associating graphs with algebraic structures was pioneered by Arthur Cayley in 1878 through the concept of Cayley graphs, which provided a graphical representation of group structures. Since then, the use of graph Theory to interpret and analyze algebraic properties has grown significantly, with many algebraic problems being re-expressed in terms of graph parameters such as topological indices. Among these developments, the concept of inverse graphs in finite groups, introduced in 2017, has gained considerable attention. Researchers have investigated various topological indices on such graphs, as seen in prior work on inverse graphs associated with finite cyclic groups. The present study extends this line of research by focusing on generalized quaternion groups. These groups exhibit regularity and a constant diameter in their corresponding inverse graphs, making them ideal for symbolic analysis of structural properties. This growing field demonstrates the deepening intersection between group Theory and graph Theory. It highlights the potential of topological indices, such as the ABC index, in revealing hidden patterns within algebraic systems.

This study aims to explore the ABC index applied to both the inverse graph of generalized quaternion groups and the non-commuting graph of finite non-abelian groups. Through theoretical and symbolic approaches, this analysis provides a comprehensive characterization of graph connectivity derived from algebraic structures, contributing to the broader application of topological indices in algebra and other scientific domains.

2. METHOD

This study is a literature review that aims to examine and analyze the application of the Atom Bond Connectivity (ABC) index on graphs derived from finite group structures, with a particular focus on inverse graphs of generalized quaternion groups and non-commuting graphs of non-abelian groups. The research adopts a qualitative and theoretical approach, investigating mathematical structures through symbolic computation rather than relying on numerical simulations or software-based analysis. The inverse graph is constructed by identifying group elements that are not self-inverse and defining connections based on whether the product of two elements equals the identity. For non-commuting graphs, vertices are defined from group elements that lie outside the center of the group, and edges are created between elements that do not commute.

The analytical process begins with the identification of the vertex set and edge set for each type of graph. Each vertex represents a group element, and the adjacency criterion is determined by the algebraic properties specific to each graph type. After constructing the graph structure, the degree of each vertex is computed, which is a crucial component in calculating the ABC index. The ABC index is a topological descriptor that quantifies the connectivity of a graph based on the degrees of adjacent vertices. Specifically, it is defined for each edge by taking the square root of the ratio involving vertex degrees, which is then summed over all edges in the graph.

To establish the ABC index for both graph types, the study derives several mathematical lemmas and theorems. These are formulated based on the known structural properties of the generalized quaternion group and the non-abelian group under study. Each result is proven through logical deduction and algebraic manipulation, ensuring the validity of the conclusions. The theoretical nature of this research enables a deep exploration of the interaction between group Theory and graph Theory, and the derivation of general formulas for the ABC index provides valuable insights into the structural characteristics of the graphs. Importantly, the entire process is performed manually, highlighting the symbolic and deductive nature of the investigation. This approach not only reinforces the theoretical understanding of algebraic graph representations but also contributes to the broader application of topological indices in pure mathematics.

3. RESULTS AND DISCUSSION

This study discusses the Atom Bond Connectivity (ABC) index on the inverse graph of the generalized quaternion group and the ABC index of the non-commuting graph of finite non-abelian groups. The ABC index, short for Atom Bond Connectivity, was first introduced by Estrada and his colleagues [6] as a new topological index. It is defined in Definition 3.1.

Definition 3.1. [6] Let $G = (V(G), E(G)$ be a graph, where $V(G)$ is the set of vertan edge connect vertices u and vegree of a vertex u is denoted by $d(u)$. Let vertices u and v be connected by an edge, denoted uv. Then, the Atom Bond Connectivity (ABC) index of the graph G is defined as

$$ABC(G) = \sum_{uv \in E(G)} \sqrt{\frac{d(u) + d(v) - 2}{d(u)d(v)}}.$$

The following section discusses the Atom Bond Connectivity (ABC) index on the inverse graph of the quaternion group.

Following the steps outlined in the methodology section—namely, the identification of graph structures and the determination of vertex degrees—this section presents a detailed discussion of the findings regarding the Atom Bond Connectivity (ABC) index. The focus lies on its application to two types of graphs derived from finite groups: the inverse graph of generalized quaternion groups and the non-commuting graph of finite non-abelian groups. The discussion is carried out systematically, referring to relevant definitions, lemmas, and theorems to provide a deeper understanding of the graph structures and the resulting ABC index values.

Atom Bond-Connectivity Index on the Inverse Graph of the Quaternion Group

This study examines the Atom Bond Connectivity (ABC) index on the inverse graph constructed from the quaternion group. By computing the ABC index, a characterization of the connectivity pattern among group elements based on their inverses is obtained, helping to identify unique structural properties of quaternion groups through a graph-theoretic approach. The definition of the quaternion group is explained in Definition 3.2.

Definition 3.2. [8] A quaternion group is a group generated by two elements $\langle a, b\rangle$. A generalized quaternion group of order $2n$ is defined as

$$Q_{2n} = \langle\, a, b \mid a^{2n-1} = e, a^{2n-2} = b^2, b^{-1}ab = a^{-1}\, \rangle.$$

With the structure of the generalized quaternion group established in Definition 3.2, we proceed by defining the inverse graph, which will serve as a key tool in exploring group-based graph constructions. Definition 3.3 defines the inverse graph.

Definition 3.3. [3] Let $(\Gamma,*)$ be a finite group and S is the set of elements in Γ be the set of elements that are not self-inverse. Then, the inverse graph $G_S(\Gamma)$ is a graph in which each vertex represents an element of Γ. Two distinct vertices u and v are adjacent if and only if $u * v$ or $v * u \in S$.

Based on the definition of the inverse graph provided above, we can now analyze the degree of each vertex in the graph through Lemma 3.1.

Lemma 3.1.[8] Every vertex v in $G_S(Q_n)$ has the same degree, which is $\deg(v) = 2^n - 2$. The total number of edges in $G_S(Q_n)$ is given by

$$\mid E\big(G_S(Q_n)\big) \mid = \frac{2^n(2^n - 2)}{2}.$$

After determining the uniform degree of vertices and the total edges in Lemma 3.1, we proceed by identifying the self-inverse elements in the group, as summarized in Lemma 3.2, which is essential for calculating the ABC index. To compute the ABC index, it is necessary to identify the self-inverse elements in the generalized quaternion group. This is summarized in Lemma 3.2.

Lemma 3.2. [8] In the generalized quaternion group Q_{2^n}, only two elements are self-inverse, namely the identity e and element $a^{2^{n-2}}$. Thus, the set S, the elements that are not self-inverse, is

$$S = Q_{2^n} \setminus \{e, a^{2^{n-2}}\}.$$

Using the results of Lemma 3.1 and Lemma 3.2, we can formulate a theorem that provides the general form of the ABC index for the inverse graph of the generalized quaternion group. It is shown in Theorem 3.1.

Theorem 3.1. Let $G_S(Q_{2^n})$ be the inverse graph of the generalized quaternion group Q_{2^n}, with $n > 2$. Then the Atom Bond Connectivity index of this graph is given by

$$ABC(G) = 2^{n-1}\sqrt{2^{n+1} - 6}.$$

Proof 3.1. Based on Definition 3.1, The ABC index is defined as

$$ABC(G) = \sum_{uv \in E(G)} \sqrt{\frac{d(u) + d(v) - 2}{d(u)d(v)}}.$$

Hence,

$$\begin{aligned} ABC(G) &= \sum_{\{uv\} \in E(G)} \sqrt{\frac{(2^n - 2) + (2^n - 2) - 2}{(2^n - 2)^2}} \\ &= \frac{2(2^n - 2)}{2}\sqrt{\frac{2(2^n - 2) - 2}{(2^n - 2)^2}} \\ &= \frac{2(2^n - 2)}{2}\sqrt{\frac{2^{n+1} - 6}{2^n - 2}} \\ &= 2^{n-1}\sqrt{2^{n+1} - 6}. \end{aligned}$$

Example 3.1. Let $G_S(Q_{2^n})$ be the inverse graph of the generalized quaternion group Q_{2^n}, with $n = 2$. Then, The Atom Bond Connectivity index of this graph is given by

$$\begin{aligned} ABC(G) &= \sum_{\{uv\} \in E(G)} \sqrt{\frac{(2^2 - 2) + (2^2 - 2) - 2}{(2^2 - 2)^2}} \\ &= \frac{2(2^2 - 2)}{2}\sqrt{\frac{2(2^2 - 2) - 2}{(2^2 - 2)^2}} \\ &= \frac{2(4 - 2)}{2}\sqrt{\frac{2(2) - 2}{2^2}} \\ &= \frac{4}{2}\sqrt{\frac{4 - 2}{4}} \end{aligned}$$

$$= 2\sqrt{\frac{2}{4}}$$
$$= 2\sqrt{0.5}$$
$$= 2\sqrt{\frac{2}{2}}$$
$$= 2\sqrt{2}.$$

Example 3.2. Let $G_S(Q_{2^n})$ be the inverse graph of the generalized quaternion group Q_{2^n}, with $n = 2$. Then the Atom Bond Connectivity index of this graph is given by

$$ABC(G) = \sum_{\{uv\}\in E(G)} \sqrt{\frac{(2^3-2)+(2^3-2)-2}{(2^3-2)^2}}$$
$$= \frac{2(2^3-2)}{2}\sqrt{\frac{2(2^3-2)-2}{(2^3-2)^2}}$$
$$= \frac{2(6)}{2}\sqrt{\frac{12-2}{36}}$$
$$= 6\sqrt{\frac{10}{36}}$$
$$= 6\sqrt{\frac{10}{6}}$$
$$= 4\sqrt{10}.$$

Table 3.1 is The Atom-Bond Connectivity (ABC) indices given for the inverse graphs of $G_s(Q_{2^n})$ with $n = 2,3,4$.

Table 3.1. Atom Bond Connectivity.

Atom Bond connectivity indices for $G_s(Q_{2^n})$; $n = 2,3,4$			
Index	$n = 2$	$n = 3$	$n = 4$
$ABC(G_s(Q_{2^n}))$	$2\sqrt{2}$	$4\sqrt{10}$	$8\sqrt{26}$

Table 3.1 presents the Atom Bond Connectivity (ABC) indices for the inverse graphs of the generalized quaternion groups Q_{2^n} for selected values of n. These values are computed based on the uniform degree of the vertices and the total number of edges, as described in Lemma 3.1 and Lemma 3.2. It is evident that as n increases, the ABC index also increases, indicating the growing complexity and connectivity of the graph structure. The ABC index thus serves as a useful quantitative measure of how the structure of Q_{2^n} evolves with increasing group order.

Atom Bond-Connectivity Index of the Non-Commuting Graph for Finite Non-Abelian Groups

Topological indices are important tools in representing the graph structure of a system, including non-commuting graphs of finite non-abelian groups. One commonly used index is the Atom Bond Connectivity (ABC) index, which represents the graph's topological structure based on vertex degrees. This section discusses the calculation and characteristics of the ABC index for non-commuting graphs associated with finite non-abelian groups, specifically on binary dihedral groups BD_{4n}. To define the non-commuting graph, we first introduce the concept of the center of a group in Definition 3.4.

Definition 3.4. [9] Let G be a group. The set of all elements in G that commute with every element in the group is called the center of the group and is denoted by

$$Z(G) = \{\, g \in G \mid gx = xg \; untuk\; semua \; x \in G \,\}.$$

By Definition 3.4, we are now introduce the non-commuting graph, whose construction relies on elements that do not commute with others in the group. The non-commuting graph is defined in Definition 3.5 as follows.

Definition 3.5. [10] The non-commuting graph $\Gamma(G)$ is a simple undirected graph formed from a non-abelian group G with the vertex set consisting of all elements not in the center G is $G \backslash Z(G)$. Two vertices a and b are connected by an edge if $ab \neq ba$, a and b are not commutative.

Several known structural properties of non-commuting graphs are summarized in the following propositions.

Proposition 3.1. All non-commuting graphs formed from non-abelian groups have a constant diameter of 2, indicating that any two vertices are connected by at most two steps.

Proposition 3.2. Every vertex in ΓG has the same eccentricity, which is 2, classifying it as a self-centered graph where all vertices are equally central.

Proposition 3.3. The non-commuting graph of finite subgroups of $SL(2, \mathbb{C})$, takes the form of a complete multipartite graph with partitions determined by the group, such as BD_{4n}, BT_{24}, BO_{48}, BI_{120}.

Utilizing the structural properties and edge classifications of the non-commuting graph, we can now state the Theorem 3.2 for its ABC index.

Theorem 3.2. Let $\Gamma_{BD_{4n}}$ be the non-commuting graph of group BD_{4n}. Then, The ABC index is

$$ABC(\Gamma_{BD_{4n}}) = \sqrt{12n(n-1)^2} + \frac{n\sqrt{8n-10}}{2}$$

$$ABC_4\left(\Gamma_{BD_{4n}}\right)=\frac{\sqrt{20n^3-28n^2+6n}}{3n-2}+\frac{\sqrt{2n^2(2n-1)(6n-7)}}{3n-2}.$$

Proof 3.2. Based on the edge classification in Table 3.2 into the ABC index calculation ABC_4 obtained as follows

$$\begin{aligned}ABC\left(\Gamma_{BD_{4n}}\right)&=\frac{(4n^2-4n)\sqrt{6n-6}}{8n(n-1)}+\frac{4n(n-1)\sqrt{8n-10}}{8(n-1)}\\&=\frac{(4n^2-4n)\sqrt{8(3n-2)(n-1)-2}}{8n(n-1)(3n-2)}\\&\quad+(4n^2-4n)\sqrt{\frac{4(n-1)(3n-2)+8n(n-1)-2}{16n(n-1)^2(6n-4)}}\\&=\sqrt{\frac{20n^3-28n^2+6n}{3n-2}}+\sqrt{\frac{2n^2(2n-1)(6n-7)}{3n-2}}.\end{aligned}$$

To facilitate the calculation of the ABC index, the edges of the graph are partitioned based on the degree of the connected vertices. These partitions are summarized in Table 3.2.

Table 3.2. Edges partition $\Gamma_{BD_{4n}}$ for each $u\sim w\in E(\Gamma_{BD_{4n}})$.

(d_v,d_w)	(S_v,S_w)	Edges Count
$((4n-4),(4n-4))$	$((4n-4)(3n-2),$ $(4n-4)(3n-2))$	$2n(n-1)$
$(2n,(4n-4)$	$(8n(n-1),$ $(4n-4)(3n-2)$	$4n(n-1)$

According to Table 3.2, the pair (d_v,d_w) classifies an edge based on the degree of each vertex, while (S_v,S_w) classifies an edge based on the number of neighboring degrees of the edge's end vertices.

Example 3.3. Based on the edge classification in Table 3.2 into the ABC index calculation for BT_{24} obtained as follows

$$\begin{aligned}ABC\left(\Gamma_{BT_{24}}\right)&=\frac{(4(6)^2-4(6))\sqrt{6(6)-6}}{8(6)(6-1)}+\frac{4(6)(6-1)\sqrt{8(6)-10}}{8(6-1)}\\&=\frac{(144-24)\sqrt{36-6}}{(8)(6)(5)}+\frac{120\sqrt{48-10}}{8(5)}\\&=\frac{120\sqrt{30}}{240}+\frac{120\sqrt{38}}{40}\\&=\frac{\sqrt{30}}{2}+3\sqrt{38}.\end{aligned}$$

4. CONCLUSION

This research presents a comprehensive analysis of the Atom Bond Connectivity (ABC) index in the context of inverse graphs of generalized quaternion groups and non-commuting graphs of non-abelian finite groups. The results demonstrate that the ABC index effectively reflects the underlying structural properties of these group-based graphs. For inverse graphs, the uniform degree of vertices simplifies the computation of the index, while in non-commuting graphs, the multipartite structure and self-centered characteristics highlight the symmetry and complexity of non-abelian groups. This research affirms the utility of topological indices in abstract algebraic contexts and open pathways for further studies, such as exploring other topological indices on different group structures or applying these insights in chemical graph theory and network analysis.

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